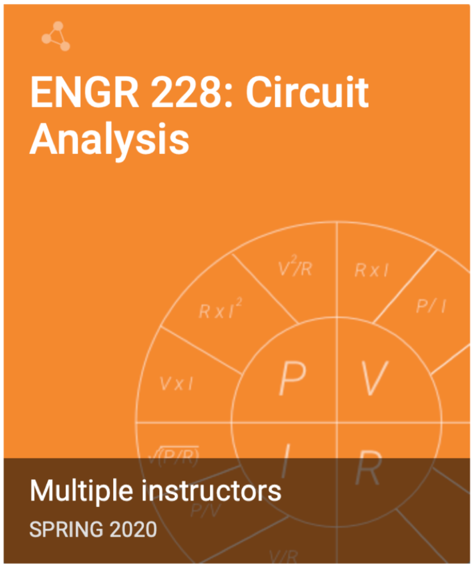


Chapter 6.8

Critically and Under-damped RLC Circuits

Engr228 - Circuit Analysis

Spring 2020

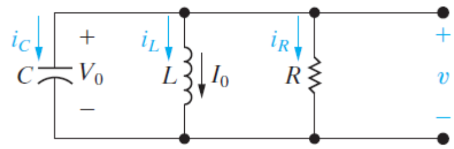
Dr Curtis Nelson

Section 6.8 Objective

- Be able to determine the natural response of critically and under-damped parallel RLC circuits.

Source-Free Parallel RLC Circuits

We will first study the natural response of second-order circuit by looking at a source-free parallel RLC circuit:



$$i_R + i_L + i_C = 0$$

$$\frac{v(t)}{R} + \frac{1}{L}\int v(t)dt + C\frac{dv(t)}{dt} = 0$$

$$C\frac{d^2v(t)}{dt^2} + \frac{1}{R}\frac{dv(t)}{dt} + \frac{1}{L}v(t) = 0$$

⇨ Second-order Differential equation

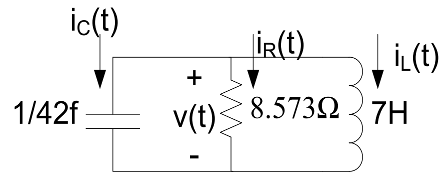
Equations for Analysing the Natural Response of Parallel *RLC* Circuits

Characteristic equation	$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$
Neper, resonant, and damped frequencies	$\alpha = \frac{1}{2RC} \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Roots of the characteristic equation	$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
$\alpha^2 > \omega_0^2$: overdamped	$v(t) = A_1e^{s_1t} + A_2e^{s_2t}, \text{t} \geq 0$ $v(0^+) = A_1 + A_2 = V_0$ $\frac{dv(0^+)}{dt} = s_1A_1 + s_2A_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$
$\alpha^2 < \omega_0^2$: underdamped	$v(t) = B_1e^{-\alpha t}\cos\omega_d t + B_2e^{-\alpha t}\sin\omega_d t, \text{t} \geq 0$ $v(0^+) = B_1 = V_0$ $\frac{dv(0^+)}{dt} = -\alpha B_1 + \omega_d B_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$
$\alpha^2 = \omega_0^2$: critically damped	$v(t) = D_1te^{-\alpha t} + D_2e^{-\alpha t}, \text{t} \geq 0$ $v(0^+) = D_2 = V_0$ $\frac{dv(0^+)}{dt} = D_1 - \alpha D_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$

(Note that the equations in the last three rows assume that the reference direction for the current in every component is in the direction of the reference voltage drop across that component.)

Critically Damped Case ($\alpha = \omega_0$)

Find v(t) in the circuit at the right.



Given initial conditions:
$v_c(0) = 0$, $i_L(0) = -10A$

$$\alpha = \frac{1}{2RC} = \omega_0 = \frac{1}{\sqrt{LC}} = 2.45$$

Critically damped when $\alpha = \omega_0$ $\qquad s_1 = s_2 = -2.45$

The complete solution is of the form:

$$v(t) = A_1 t e^{st} + A_2 e^{st}$$

Critically Damped Case - continued

Use initial conditions to find A_1 and A_2

From $v_c(0) = 0$ at t = 0:

$$v(0) = 0 = A_1(0)e^0 + A_2 e^0 = A_2$$

Therefore $A_2 = 0$ and the solution is reduced to $v(t) = A_1 t e^{-2.45t}$

Find A_1 from KCL at t = 0:

$$i_R + i_L + i_C = 0$$

$$\frac{v(0)}{R} + (-10) + C\left.\frac{dv(t)}{dt}\right|_{t=0} = 0$$

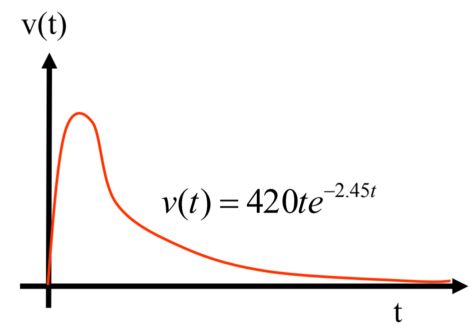
$$\frac{0}{R} + (-10) + \frac{1}{42}\left(A_1 t(-2.45)e^{-2.45t} + A_1 e^{-2.45t}\right)\Big|_{t=0} = 0$$

$$-10 + \frac{1}{42}(A_1) = 0$$

Critically Damped Case - continued

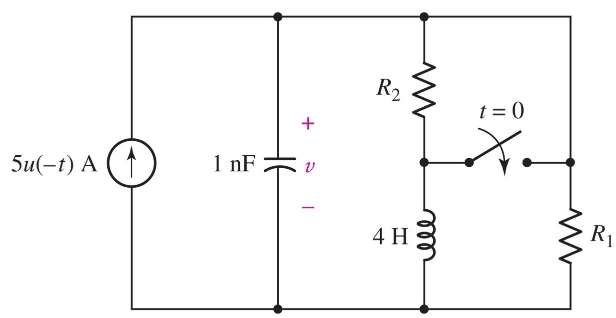
Solving the equation: $A_1 = 420$
The solution is:

$$v(t) = 420te^{-2.45t}V$$



Critically Damped Example

Find R_1 such that the circuit is critically damped for $t > 0$ and R_2 such that $v(0)=2V$.



Answer: $R_1 = 31.63\ k\Omega$, $R_2 = 0.4\Omega$

Underdamped Case ($\alpha < \omega_0$)

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

For the underdamped case, the term inside the bracket will be negative and s will be a complex number.

Define $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

Then $s_{1,2} = -\alpha \pm j\omega_d$

$$v(t) = A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{(-\alpha - j\omega_d)t}$$
$$v(t) = e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t})$$

Underdamped Case - continued

$$v(t) = e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t})$$

Using Euler's Identity $e^{j\theta} = \cos\theta + j\sin\theta$

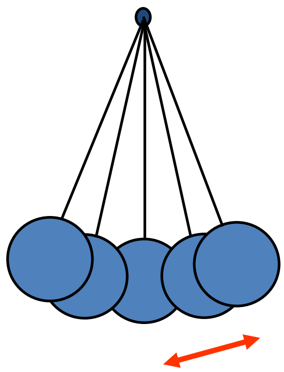
$$v(t) = e^{-\alpha t}(A_1 \cos\omega_d t + jA_1 \sin\omega_d t + A_2 \cos\omega_d t - jA_2 \sin\omega_d t)$$
$$v(t) = e^{-\alpha t}((A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t)$$
$$v(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)$$

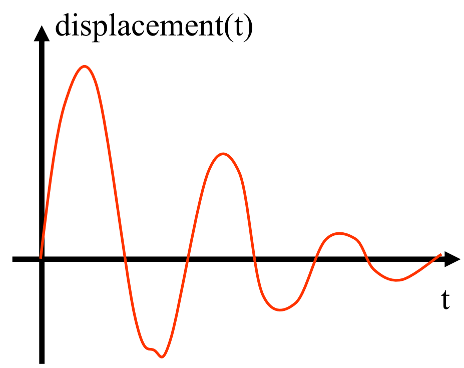
Looking at the magnitude:

$$v(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)$$

Mechanical Analogue

A pendulum is an example of an underdamped second-order mechanical system.

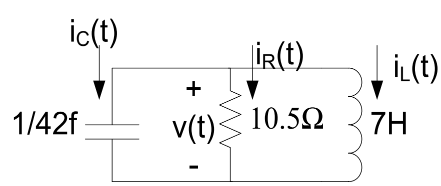




Underdamped Case - Example

Find v(t) in the circuit at the right.

Given initial conditions:
$v_c(0) = 0$, $i_L(0) = -10A$



$$\alpha = \frac{1}{2RC} = 2 \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{6}$$

$\alpha < \omega_0$ therefore, this is an underdamped case

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{2}$$

v(t) is of the form: $v(t) = e^{-2t}(B_1 \cos\sqrt{2}t + B_2 \sin\sqrt{2}t)$

Equations for Analysing the Natural Response of Parallel *RLC* Circuits

Characteristic equation	$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$
Neper, resonant, and damped frequencies	$\alpha = \frac{1}{2RC} \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$
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$\alpha^2 > \omega_0^2$: overdamped	$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}, t \geq 0$ $v(0^+) = A_1 + A_2 = V_0$ $\frac{dv(0^+)}{dt} = s_1 A_1 + s_2 A_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$
$\alpha^2 < \omega_0^2$: underdamped	$v(t) = B_1 e^{-\alpha t}\cos \omega_d t + B_2 e^{-\alpha t}\sin \omega_d t, t \geq 0$ $v(0^+) = B_1 = V_0$ $\frac{dv(0^+)}{dt} = -\alpha B_1 + \omega_d B_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$
$\alpha^2 = \omega_0^2$: critically damped	$v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}, t \geq 0$ $v(0^+) = D_2 = V_0$ $\frac{dv(0^+)}{dt} = D_1 - \alpha D_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$

(Note that the equations in the last three rows assume that the reference direction for the current in every component is in the direction of the reference voltage drop across that component.)

Underdamped Case - continued

Use initial conditions to find B_1 and B_2

From $v_c(0) = 0$ at t = 0:

$$v(0) = e^0(B_1 \cos 0 + B_2 \sin 0) = B_1$$

Therefore $B_1 = 0$ and the solution is reduced to

$$v(t) = e^{-2t}(B_2 \sin \sqrt{2}t)$$

Find B_2 from KCL at t = 0:

$$i_R + i_L + i_C = 0$$

$$\frac{v(0)}{R} + (-10) + C\left.\frac{dv(t)}{dt}\right|_{t=0} = 0$$

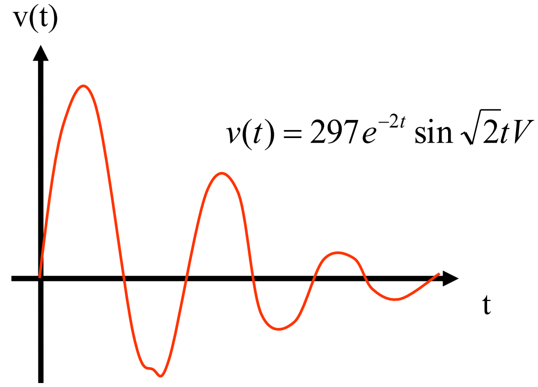
$$\frac{0}{R} + (-10) + \frac{1}{42}\left.\left(\sqrt{2}B_2 e^{-2t}\cos\sqrt{2}t - 2B_2 e^{-2t}\sin\sqrt{2}t\right)\right|_{t=0} = 0$$

$$-10 + \frac{1}{42}(\sqrt{2}B_2) = 0$$

Underdamped Case - continued

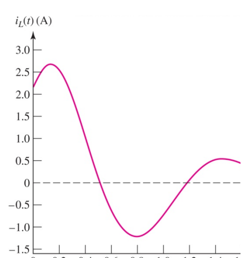
Solving: $B_2 = 210\sqrt{2} = 297$

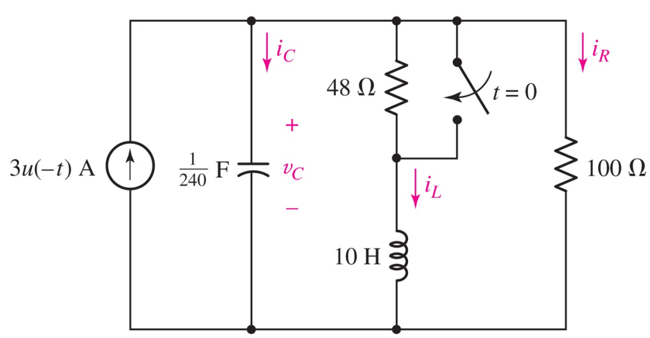
The solution is: $v(t) = 297e^{-2t}\sin\sqrt{2}tV$



Underdamped Example

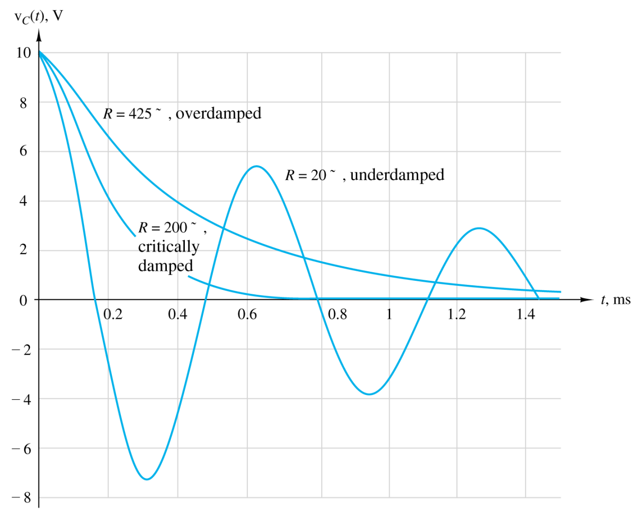
Find i_L for $t > 0$.





$i_L = e^{-1.2t}(2.027 \cos 4.75t + 2.561 \sin 4.75t)\ A$

Summary of Transient Responses



Section 6.8 Summary

- Showed how to determine the natural response of critically and under-damped parallel RLC circuits.